On the number of returns for a linear diffusion

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§ 1. Introduction

CIESIELSKI-TAYLOR [2] and RAY [3] obtained theorems of the so-called iterated logarithm type concerning the sojourn time of the multi-dimensional Brownian path within a small sphere. The classical law of the iterated logarithm is concerned with the asymptotic behaviours of a stochastic process as the time parameter varies. However, the above cited authors discussed the fluctuations of the sojourn times in terms of the points of state space. Their investigation was originated from a problem about the Hausdorff measure of the path. For one-dimensional diffusion process, there is no problem about the sojourn time since TROTTER [4] proved the existence of the "local time", i.e., the continuous density function of the sojourn time.

In order to obtain the result for the sojourn time, RAY [3] considered the number of returns from a point to another before crossing a fixed one and discussed the order for its fluctuation. In this note we are concerned with the same problem for a non-singular one-dimensional diffusion process. The form of our theorem is stated at the first part of section 3.

§ 2. Preliminaries

Let us consider a one-dimensional diffusion with state interval $S=[0, \infty)$; that is, a temporally homogeneous Markov process with continuous sample paths $w\,;\, t \rightarrow X(t) \in S$, satisfying the strong Markov property. We assume that, for every point ξ and η in S, the communicative relation, i.e.

$$(1) \qquad P_{\xi}\{\sigma(\eta)<+\infty\}>0\ , \qquad \sigma(\eta)=\min\{t: X(t)=\eta\}\ ,$$

where $P_{\xi}\{\cdot\}$ denotes the probability governing the path starting at ξ. Then there exist the associated scale $s(\xi)$ and the speed measure $m(d\xi)$ in $(0, +\infty)$ which were introduced by W. FELLER. The probabilistic significance of the scale $s(\xi)$ is given by

$$(2) \qquad P_{\xi}\{\sigma(\alpha)<\sigma(\beta)\}=\frac{s(\beta)-s(\xi)}{s(\beta)-s(\alpha)}\ , \qquad \alpha<\xi<\beta\ .$$

* The result in this note was reported in October, 1964 at the meeting of Mathematical Society of Japan. The present address of the author is as follows; Department of Mathematics, Ochanomizu University, Otsuka, Tokyo.

We shall make a further assumption. Choose an arbitrary point B in $(0, +\infty)$ and fix it. Suppose that for every ξ with $0 \leqq \xi \leqq B$,

$$(3) \qquad E_\xi\{\sigma(0)\} < +\infty \text{ [1]} \quad \text{and} \quad E_\xi\{\sigma(B)\} < +\infty \;.$$

Consider now the monotone decreasing sequence $\{a_n;\, n \geqq 1\}$ defined by using the scale $s(\xi)$,

$$(4) \qquad s(0) = 0 \;, \quad s(a_n) = b/n \;, \quad s(B) = b \;.$$

For a fixed n, let t_ν $(\nu = 0, 1, 2, \cdots)$ be the successive passage times at which $X(t, w)$ runs through the points $a_n, a_{n+1}, a_n, a_{n+1}, \cdots\cdots$. To be precise, set

$$t_0(w) = 0 \;,$$

$$t_{2\nu+1}(w) = \inf\{\tau;\, \tau > t_{2\nu}(w),\, X(\tau, w) > a_n\} \;,$$

$$t_{2\nu}(w) = \inf\{\tau;\, \tau > t_{2\nu-1}(w),\, X(\tau, w) < a_{n+1}\} \;.$$

By assumption (3), t_ν is defined for all positive integers ν.

Define further

$$X_\nu(\tau, w) = X(\tau, w) \;, \qquad t_\nu(w) \leqq \tau < t_{\nu+1}(w) \;,$$

$$N_n(w) = \max\{\nu;\, t_{2\nu}(w) < \sigma(B)\} \;,$$

that is, N_n is the number of returns from a_n to a_{n+1} before the first passage across B.

From (2) we have

$$(5) \qquad P_{a_n}\{\sigma(a_{n+1}) < \sigma(B)\} = \frac{s(B) - s(a_n)}{s(B) - s(a_{n+1})} = \left(1 - \frac{1}{n^2}\right) .$$

Using the strong Markov property and (5) successively, we deduce that

$$(6) \qquad P_0\{\, N_n \geqq k \,\} = \left(1 - \frac{1}{n^2}\right)^k .$$

Let $m > n$. Let $N_{n,m,\nu}$ with $\nu = 0, 1, 2, \cdots$ be the number of returns from a_m to a_{m+1} of the process $X_{2\nu}$. Then it is clear that

$$(7) \qquad N_m = \sum_{\nu=0}^{N_n} N_{n,m,\nu} \;.$$

As in [3], the variables N_n $(n = 1, 2, \cdots)$ turn out to be Markovian, that is, for given N_n, N_m is independent of N_k, $k < n$.

The distribution of $N_{n,m,\nu}(m > n)$ is given from the analogous argu-

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[1] E_ξ is the expectation with respect to P_ξ.

ment used above to derive (6):

$$P_{a_m}\{\sigma(a_{m+1})<\sigma(a_n)\} = \left(1+\frac{1}{m}\right)\left(\frac{m-n}{m-n+1}\right),$$

$$P_0\{N_{n,m,0}\geqq k\} = \left(1+\frac{1}{m}\right)^k\cdot\left(\frac{m-n}{m-n+1}\right)^k.$$

For $\nu>0$, $N_{n,m,\nu}\geqq k>1$ if and only if (i) $N_{n,m,\nu}\geqq 1$ and (ii) $(k-1)$ independent copies of the process starting at a_m reach a_{m+1} before a_n. Thus for $\nu>0$,

$$P_0\{N_{n,m,\nu}\geqq k\} = P_{a_{n+1}}\{N_{n,m,\nu}\geqq 1\}\cdot\left(1+\frac{1}{m}\right)^{k-1}\left(\frac{m-n}{m-n+1}\right)^{k-1}$$
$$= \frac{1}{n+1}\cdot\left(\frac{m+1}{m-n+1}\right)^k\cdot\left(\frac{m-n}{m}\right)^{k-1}, \qquad k\geqq 1.$$

Hence the generating function is computed as follows:

$$E_0\{\sigma^{N_{n,m,\nu}}\} = \frac{n}{n+(m+1)(m-n)(1-\sigma)}, \qquad \nu=0,$$
$$=1-\frac{(1-\sigma)A}{1-\sigma B}. \qquad \nu>0,$$

where

$$A=\frac{m+1}{n+1}\cdot\frac{1}{m-n+1}, \qquad 1>B=\frac{(m+1)(m-n)}{m(m-n+1)}>0. \tag{8}$$

Finally since the $N_{n,m,\nu}$ are independent of N_n, (7) implies

$$E_0\{\sigma^{N_m}|N_n\} = \left[1+\frac{m+1}{n}(m-n)(1-\sigma)\right]^{-1}\left[1-\frac{(1-\sigma)A}{1-\sigma B}\right]^{N_n}. \tag{9}$$

§ 3. Theorem and the Proof

Next we study the order for the fluctuation of Markov chain N_n as $n\to\infty$.

Theorem. *For any positive monotone non-decreasing function g, the proposition*

$$P_0\{N_n>n^2g(n) \text{ for infinitely many } n\}=0 \text{ or } 1$$

holds according as the integral

$$\int^\infty \frac{1}{t}g(t)e^{-g(t)}dt \tag{10}$$

is convergent or divergent.

Therefore we easily obtain

$$P_0\left\{\limsup_{n\to\infty}\frac{N_n}{n^2\log\log n}=1\right\}=1\,.$$

Remark. This theorem is derived from the scale only and is independent of the speed measure.

Proof. Set

$$x=\sqrt{1-B}(\sqrt{A}-\sqrt{1-B})\,,\qquad \sigma=(1+x)^{-1}\,.$$

Then by (8) we may assume that $x>0$ and so $\sigma<1$. If $N>0$, and $m>n$, we get the following estimate with the help of (9):

$$\begin{aligned}
P_0\{N_m<N\,|\,N_n=N\}&\leqq\sigma^{-N}E_0\{\sigma^{N_m}|N_n=N\}\\
&=\sigma^{-N}\left[1+\frac{m+1}{n}(m-n)(1-\sigma)\right]^{-1}\left[1-\frac{(1-\sigma)A}{1-\sigma B}\right]^N\\
&<(1+x)^N\left[1-\frac{xA}{1+x-B}\right]^N\\
&<\exp\left\{xN-\frac{xAN}{1+x-B}\right\}\\
&=\exp\{-N(\sqrt{A}-\sqrt{1-B})^2\}\\
&<1-\delta\,.
\end{aligned}$$

Combining the Markovian character with the above estimate, we infer that

$$\begin{aligned}
P_0\{N_m\geqq y\}&\geqq P_0\{\max_{n\leqq k<m}N_k\geqq y,\ N_m\geqq y\}\\
&=\sum_{k=n}^{m-1}P_0\{\max_{n\leqq j<k}N_j<y,\ N_k\geqq y,\ N_m\geqq y\}\\
&\geqq\delta\cdot\sum_{k=n}^{m-1}P_0\{\max_{n\leqq j<k}N_j<y,\ N_k\geqq y\}\\
&=\delta\cdot P_0\{\max_{n\leqq k<m}N_k\geqq y\}\,.
\end{aligned}\tag{11}$$

Let us define a sequence $\{l_k\}$ as follows:

$$l_{k+1}^2=l_k^2(1+1/g(l_k))\,,\tag{12}$$

where l_1 is chosen sufficiently large enough to satisfy the condition $l_k\to\infty$ as $k\to\infty$.

Since $g(l)$ is monotone increasing, we have, by (12),

$$\int_{l_1}^{\infty}\frac{1}{l}g(l)e^{-g(l)}dl\geqq\sum_k\frac{l_{k+1}-l_k}{l_{k+1}}g(l_{k+1})e^{-g(l_{k+1})}$$

$$= \sum_k \frac{l_{k+1}^2 - l_k^2}{l_{k+1}(l_{k+1}+l_k)} g(l_{k+1}) e^{-g(l_{k+1})}$$

$$\geqq \sum_k \frac{l_k^2 \cdot (g(l_k))^{-1}}{2 l_{k+1}^2} g(l_{k+1}) e^{-g(l_{k+1})}$$

$$\geqq \frac{1}{2} \cdot \frac{1}{1+1/g(l_1)} \sum_k e^{-g(l_{k+1})} .$$

But, for large k, it follows from (11), (6) and (12) that

$$P_0\left\{ \max_{l_k \leqq n < l_{k+1}} \frac{N_n}{n^2 g(n)} \geqq 1 \right\} \leqq P_0\left\{ \max_{l_k \leqq n < l_{k+1}} N_n \geqq l_k^2 g(l_k) \right\}$$

$$\leqq \delta^{-1} \cdot P_0\{ N_{l_{k+1}} \geqq l_k^2 g(l_k) \}$$

$$= \delta^{-1} \left(1 - \frac{1}{l_{k+1}^2}\right)^{l_k^2 g(l_k)}$$

$$\leqq \delta^{-1} \cdot \exp\left\{ - \frac{l_k^2}{l_{k+1}^2} g(l_k) \right\}$$

$$\leqq \delta^{-1} \cdot \exp\left\{ - \left(1 - \frac{1}{g(l_k)}\right) g(l_k) \right\}$$

$$= \frac{e}{\delta} \cdot e^{-g(l_k)} .$$

Thus if the integral (10) converges, we can conclude by the Borel-Cantelli lemma that

$$P_0\{ N_n > n^2 g(n) \text{ infinitely often} \} = 0 .$$

We turn to the case in which the integral (10) diverges. For the sequence $\{l_k\}$ defined in (12), we have

$$\int_{l_1}^{\infty} \frac{1}{l} g(l) e^{-g(l)} dl \leqq \sum_k \frac{l_{k+1} - l_k}{l_k} g(l_k) e^{-g(l_k)}$$

$$= \sum_k \frac{l_{k+1}^2 - l_k^2}{l_k(l_{k+1}+l_k)} g(l_k) e^{-g(l_k)}$$

$$\leqq \frac{1}{2} \sum_k e^{-g(l_k)} .$$

Hence $e^{-g(l_k)}$ is the general term of a divergent series. By a general result on infinite series, there exists a subsequence $\{n_k\}$ of $\{l_k\}$ with the property that, as $k \to \infty$

$$n_{k+1} - n_k \to \infty , \qquad \sum_k e^{-g(n_k)} = \infty$$

We denote by A_k the event

$$\{ N_{n_k} \geqq n_k^2 g(n_k) \} .$$

Then applying (6), we have

$$P_0\{A_k\} \sim \exp\{-g(n_k)\}\ , \qquad k \longrightarrow \infty\ .$$

The events $\{A_k\}$ $(k=1, 2, \cdots)$ are of course not independent. However, since the variables N_n form a Markov chain with parameter n, the choice of the sequence $\{n_k\}$ enables us to verify that the conditions of CHUNG-ERDÖS [1] hold. These observations lead to the proof of the latter part of our theorem.

References

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